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LETTER TO THE EDITOR

A symmetry group of a Thue–Morse quasicrystalJean-Pierre Gazeau^{†§} and Jacek Miękisz^{‡||}[†] Laboratoire de Physique Théorique de la Matière Condensée, Université Paris 7 - Denis Diderot, 2, place Jussieu, F-75251 Paris Cedex 05, France[‡] Institute of Applied Mathematics and Mechanics, University of Warsaw, ul Banacha 2, 02-097 Warsaw, Poland

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Abstract. We present a method of coding general self-similar structures. In particular, we construct a symmetry group of a one-dimensional Thue–Morse quasicrystal, i.e. of a nonperiodic ground state of a certain translation-invariant, exponentially decaying interaction.

A symmetry group of a three-dimensional crystal consists of lattice translations, rotations, and reflections. Starting from any point of a crystal, we can reach any other point, successively applying different elements of the symmetry group of the crystal. It was recently shown [1, 2] that certain one-dimensional quasicrystals can be built by successive applications, on one of its points, of elements of certain discrete affine semigroups. Here we describe a general method, based on ideas contained in [3, 4], of representing self-similar structures by one-sided sequences of two symbols. In particular, we construct a symmetry group of a Thue–Morse quasicrystal, i.e. of a nonperiodic ground state of a certain one-dimensional classical lattice-gas model.

In one-dimensional classical lattice-gas models, every site of the lattice \mathbb{Z} (the set of all integers) can be occupied by a particle or be empty. Configurations of such models are therefore elements of $\Omega = \{0, 1\}^{\mathbb{Z}}$, where 1 denotes the presence and 0 denotes the absence of a particle at any given lattice site. By $X(i)$ we denote the projection of X to a lattice site $i \in \mathbb{Z}$.

The Thue–Morse example. We use the Thue–Morse substitution rule to construct a configuration in Ω . We put 1 at the origin and perform alternatively to the right and to the left the following substitution: $1 \rightarrow 10, 0 \rightarrow 01$. After the first right substitution we obtain 10, which is the configuration on $[0, 1]$, then the substitution to the left gives us 0110 on $[-2, 1]$ (the substitution to the left means that we replace 1 by 01 and 0 by 10), then we obtain 01101001 on $[-2, 5]$ and so on. The effect of performing the substitution to the right or left is the same as taking a sequence of symbols already obtained, changing every 0 to 1 and every 1 to 0 and then placing the new sequence either right or left in relation to the previous sequence.

In this manner we obtain a nonperiodic configuration $X_{TM} \in \Omega$. Note that if all substitutions were performed to the right, we would obtain the well known one-sided Thue–Morse configuration: 1001011001101001... [5–7].

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Let T be a translation operator, i.e. $T : \Omega \rightarrow \Omega$, $T(X)(i) = X(i - 1)$, $X \in \Omega$. Let G_{TM} be a closure (in the product topology of the discrete topologies on $\{0, 1\}$) of the orbit of X_{TM} by translations, i.e. $G_{TM} = \{T^n(X_{TM}), n \geq 0\}^{cl}$. We call elements of G_{TM} two-sided Thue–Morse configurations. It can be shown that G_{TM} supports exactly one translation-invariant probability measure μ_{TM} on Ω [8, 9]. Such a measure is said to be uniquely ergodic and can be obtained as the limit of averaging over X_{TM} and its translates: $\mu_{TM} = \lim_{n \rightarrow \infty} (1/n) \sum_{i=1}^n \delta(T^i(X_{TM}))$, where $\delta(T^i(X_{TM}))$ is the probability measure assigning probability 1 to $T^i(X_{TM})$. It means that all two-sided Thue–Morse configurations look locally identical—any local pattern of particles appears in all of them with the same density (defined uniformly in space). It is also said that such configurations belong to the same isomorphism class.

The Thue–Morse measure is the unique ground state, i.e. a measure supported by configurations with the minimal energy density, of certain exponentially decaying, translation-invariant, four-body interaction [10–12]. Multilayer Thue–Morse superlattice heterostructures were recently made by means of the molecular beam epitaxy and were investigated by Raman scattering [13] and high resolution x-ray diffraction [14]. Nonperiodic order present in Thue–Morse configurations (and in Fibonacci configurations defined below) was investigated in [15–17].

The Fibonacci example. We repeat the above procedure using the Fibonacci substitution: $0 \rightarrow 01, 1 \rightarrow 0$. We put 0 at the origin and apply alternatively the above substitution to the right and left. We obtain the configuration $X_F \in \Omega$. If all substitutions were performed to the right we would get the right one-sided Fibonacci configuration $01001010\dots$. Denote by G_F the closure of the orbit of X_F by translations. The elements of G_F are called two-sided Fibonacci configurations. G_F supports the uniquely ergodic measure μ_F which is the unique ground state of any exponentially decaying, strictly convex, repulsive interaction and a chemical potential which fixes the density of particles to be equal to the square of the golden ratio $(2/(1 + \sqrt{5}))^2$ [18–20].

We shall now discuss a concept of self-similarity. Let $X \in \Omega$. Let us assume that there are two types of finite configurations, we denote them by s_0, s_1 , such that we can group all symbols of X into successive local configurations of these types. We then construct $Y_X \in \Omega$ in the following way. If $0 \in \mathbb{Z}$ belongs to the support of a local configuration of the s_j type, then $Y_X(0) = j$, $j = 0, 1$. Now, let $s^i \in \{s_0, s_1\}$, $i = 1, 2, \dots$ be successive types of local configurations to the right of the origin and $s^i \in \{s_0, s_1\}$, $i = -1, -2, \dots$ be successive types of configurations to the left of the origin. If $s^i = s_j$, then we define $Y_X(i) = j$.

Definition. A set of configurations, $G \subset \Omega$, is *self-similar* if for some choice of s_0, s_1 , for every $X \in G$, $Y_X \in G$. Then the grouping of symbols of X into local configurations of the type s_0 or s_1 is also called self-similar.

The following proposition has been proven in [21].

Proposition 1. If a self-similar grouping of symbols, into given local configurations s_0, s_1 , of a configuration $X \in \Omega$ from a self-similar set is unique, then X is nonperiodic.

Proof by contraposition. Let us assume that X has period p . We repeat the process of grouping symbols until $p < \max\{|c_0|, |c_1|\}$, where c_0, c_1 are local configurations of X such

that after all successive groupings they are of s_0 and s_1 type respectively and $|c_0|, |c_1|$ are lengths of their supports. Now we translate X by p lattice units. X does not change. However, c_0 and c_1 overlap with their translates. The translation therefore produces a different grouping.

The Thue–Morse example. The set G_{TM} of two-sided Thue–Morse configurations is self-similar and a corresponding self-similar grouping is the following $s_0 = 01 \rightarrow 0, s_1 = 10 \rightarrow 1$.

The Fibonacci example. The set G_F of two-sided Fibonacci configurations is self-similar and a corresponding self-similar grouping is the following $s_0 = 01 \rightarrow 0, s_1 = 0 \rightarrow 1$.

We now present a method of coding self-similar sequences.

Let $G \subset \Omega$ be a self-similar set and let $X \in G$. If $X(0) = 1$, then we set $C_X(-1) = 1$; if $X(0) = 0$, then $C_X(-1) = 0$. If $0 \in \mathbb{Z}$ belongs to the support of s_j , then $C_X(0) = j$. Now we group the symbols of X and construct Y_X . If 0 belongs to the support of s_j of Y_X , then $C_X(1) = j$. We group the symbols of Y_X , construct Y_{Y_X} and obtain $C_X(2)$. We continue this procedure infinitely many times and obtain a sequence $C_X(i), -1 \leq i < \infty$. C_X can be seen as an element of a direct product $\mathcal{W} = \otimes_{i=-1}^{\infty} Z_2$, where Z_2 is the group of two elements, 0 and 1, with the addition modulo 2 as a group action.

The Fibonacci example. The Fibonacci configurations are represented by elements of $\mathcal{W}_F = \{W \in \mathcal{W}, W(i)W(i+1) = 0, \text{ for every } i \geq -1\}$. This restriction on W 's is present in a corresponding representation of Penrose tilings [3, 4].

If $C_X \in \mathcal{W}_F$ has both infinitely many 0's and 1's, then X is a two-sided Fibonacci configuration, i.e. an element of G_F . Otherwise, X is a one-sided Fibonacci configuration. For example, (0010101010...) represents X_F and (0000000...) represents the right one-sided Fibonacci configuration.

The Thue–Morse example. It is easy to see that every element of \mathcal{W} represents either a two-sided Thue–Morse configuration, i.e. an element of G_{TM} or a one-sided Thue–Morse configuration. For example, (11001100...) represents X_{TM} , (111111...) represents the right one-sided Thue–Morse configuration (with 1 at the origin) and (101010...) represents the left one-sided Thue–Morse sequence ...01101001 (with 1 at the origin) obtained by successive applications of the Thue–Morse substitution to the left. Let us note that the representation of the last configuration has infinitely many 0's and 1's. However, we would like to represent one-sided configurations by sequences with either finitely many 0's or finitely many 1's. To achieve this, we represent Thue–Morse configurations in another way.

Observe that every Thue–Morse configuration can be obtained by successive applications of the Thue–Morse substitution either to the right or to the left. If $X(0) = 1$, then let $C'_X(-1) = 1$; if $X(0) = 0$, then let $C'_X(-1) = 0$. We put $C'_X(i) = 1$ if the i th substitution was performed to the right and $C'_X(i) = 0$ if the i th substitution was performed to the left. Such C'_X is again an element of \mathcal{W} .

Now, the right one-sided Thue–Morse configuration is still represented by (111111...) but the left one is represented by (100000...) and X_{TM} by (11010101...). It is easy to see that $C'_X(i) = (C_X(i-1) + C_X(i) + 1) \bmod 2$. $X \in G_{TM}$ if and only if C'_X has both infinitely many 0's and 1's.

Obviously, for some of the elements of \mathcal{W} , corresponding Thue–Morse configurations are related by translations, reflections or the transformation changing every 0 to 1 and every 1 to 0. We say that such Thue–Morse configurations are equivalent. Let $\mathcal{U} \subset \mathcal{W}$ be a subgroup of \mathcal{W} consisting of sequences with finitely many 1's or finitely 0's. We have the following theorem.

Theorem 1. Equivalent classes of Thue–Morse configurations are represented by elements of the coset group \mathcal{W}/\mathcal{U} .

Proof. (a) Let $X_1, X_2 \in G_{TM}$ be related by a translation T^n . We group successive symbols of X_1 and X_2 into s_0 and s_1 configurations, then we group symbols of Y_{X_1} and Y_{X_2} . We repeat grouping until, say in the i th grouping, we obtain for both X_1 and X_2 the local configurations of the same type, either s_0 or s_1 , with supports containing both the origin and n (they are related by the same translation as X_1 and X_2). It follows that $C'_{X_1}(j) = C'_{X_2}(j)$, for every $j > i$ and therefore $C'_{X_1} = C'_{X_2} + U$, where $U(j) = 0$ for $j > i$.

Conversely, let $C'_{X_1} = C'_{X_2} + U$, where $U(j) = 0$ for $j > i$. Let $c(k)$ be a local configuration obtained by successive applications of substitutions corresponding to $C'_{X_k}(l), l \leq i, k = 1, 2$. If $c(1)$ and $c(2)$ are related by a translation, then X_1 and X_2 are related by the same translation. If $c(1)$ and $c(2)$ are related by a translation and the interchanging of 0's and 1's, then X_1 and X_2 are related by the same translation and the interchanging of 0's and 1's. Observe that the first situation occurs if the absolute value of the difference of the number of 1's in $C'_{X_1}(l)$ and $C'_{X_2}(l), l \leq i$, is even, otherwise we have the second case.

(b) $X_1, X_2 \in G_{TM}$ are related by the interchanging of 0's and 1's if and only if $C'_{X_1} = C'_{X_2} + (10000\dots)$.

(c) $X_1, X_2 \in G_{TM}$ are related by the reflection around the origin if and only if $C'_{X_1} = C'_{X_2} + (01111\dots)$. \square

Let us note that the above group acts on the whole set G_{TM} of the Thue–Morse configurations. We shall now construct a symmetry group which leaves every Thue–Morse configuration invariant. This corresponds to crystal symmetries mentioned at the very beginning of the paper.

A symmetry group of the Thue–Morse configurations. The following theorem gives us the positions of particles, i.e. 1's in a Thue–Morse configuration.

Theorem 2. Let X be a Thue–Morse configuration, with 1 at the origin, and represented by $C'_X \in \mathcal{W}$. We have

$$X(i) = 1 \quad \text{iff } i = \sum_{k=0}^{\infty} (-1)^{(1+C'_X(k))} 2^k a_k \quad (1)$$

where $a_k \in \{0, 1\}, k \geq 0$ and in the sum there are finite and even numbers of nonzero terms.

Proof. Let us recall that the effect of performing the substitution to the right or to the left is the same as taking a sequence of symbols already obtained, changing every 0 to 1 and every 1 to 0 and then placing the new sequence right to or left to the previous sequence. We call such sequences blocks. If we start with 1 at the origin, then every second block

placed right to or left to the previous block begins with 1. Let $X(i) = 1$. We identify the first block containing the origin to which i belongs and call it the exterior block. If i is not at the beginning of this block, we then have to find out to which one of the two subblocks it belongs. We continue this procedure until we identify the block such that i is at the beginning of it. If the exterior block begins with 1, then its position is given by the sum in (1) with an even number of nonzero terms and then we have to identify an even number of additional subblocks; to obtain (1) we have to add to the position of the left end of the exterior block an even number of different powers of 2. In the other case, the exterior block begins with 0 and its position is given by the sum in (1) with an odd number of nonzero a_i 's, but we have to identify an odd number of additional subblocks. For example, in the right one-side Thue–Morse sequence represented by (1111111...), all exterior blocks begin with 1 at the origin. \square

Denote by S_{TM} the set of all sequences $\{a_i, i \geq 0\}$ such that $a_i = 1$ for finite and even number of i 's and otherwise $a_i = 0$. S_{TM} is a subgroup of $\otimes_{i=0}^{\infty} \mathbb{Z}_2$. It is generated by an infinite number of elements (with two successive 1's and 0's otherwise). A Thue–Morse quasicrystal is generated by successive applications of elements of S_{TM} to the particle at the origin.

More precisely, positions of particles in a two-sided Thue–Morse configuration $X \in G_{TM}$ is a subset $\mathbb{Z}_X \subset \mathbb{Z}$. The coding of X, C'_X , gives us an addition, $+_X$, in \mathbb{Z}_X , such that $(\mathbb{Z}_X, +_X)$ is a group.

For example, $\mathbb{Z}_{X_{TM}}$ is generated by $\{(-1)^{k+1}2^k : k \geq 0\}$.

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